THE EDGE EFFECT IN THE OSCILLATIONS OF ELASTIC SHELLS

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PMM Vol.24, No.5, 1960, pp. 831-843

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(Received 14 January 1960)

Many technical problems require a study of the oscillations of thin-walled elements of the shell-type under high-frequency excitation. If the wave number is sufficiently large, asymptotic expressions may be constructed to represent the form of oscillation. These expressions hold everywhere except in the regions bordering on the contour or in regions containing lines of distortion; the asymptotic expressions do not, in general, satisfy boundary conditions. Tests show that thin-walled elements fracture from fatigue most frequently near lines of distortion. This suggests the idea of seeking solutions which would satisfy all boundary conditions and conditions on the lines of distortion, and which would reduce to the asymptotic expressions when receding into the internal region. The solutions found for a bounded region are reminiscent of the solutions which described the simple edge effect in the static calculation of shells. By analogy with the static edge effect, we shall call a deviation from the asymptotic expressions near distortion lines the *dynamic edge effect*. We shall see later that such an analogy has special significance.

By a separation of the solution of the equations of shell oscillation into an asymptotic solution for an internal region and a solution describing the dynamic edge effect, we obtain an effective method for solving different dynamic problems in the theory of plates and shells. The method may be applied to the spectrum analysis of natural oscillations, as well as to the estimation of stresses near a line of distortion for forced oscillations. The higher the order of oscillation frequencies the smaller the error*. In this connection, the asymptotic method successfully

* At the same time it must be kept in mind that with high-frequency

complements the linear variational methods usually applied for the solution of shell problems, methods which give results only for low frequencies and modes of oscillations.

The concept of a dynamic edge effect was introduced in [1]. Here, the concept was applied successfully to determine the frequencies and the natural modes of oscillation of rectangular plates, as well as to estimate the strength of plates under forced oscillations with a continuous excitation spectrum. The asymptotic method was applied in [2] to the study of the spectra of natural oscillations of different types of plates and the results were compared with those obtained by other methods.

The theory of the dynamic edge effect is given below for shells, together with a classification of the different types of edge effect for shells having a positive, zero or negative Gaussian curvature; also included are methods of calculation of the characteristics of the edge effect near lines of distortion according to the well-known theory for an internal region. The version of the equations of shell theory for a state with large exponents of variation was employed, and extensive use has been made of the terminology of [3].

1. Basic assumptions. We consider a thin shell undergoing free elastic oscillations of sufficiently small amplitude. We refer the middle surface of the shell to an orthogonal system of curvilinear coordinates α, β, and assume that the lines of distortion γ (for example, the contour boundaries or the axes of stiffening elements) coincide with lines α = const or β = const. We limit consideration to those modes of oscillation which correspond to large values of the exponent of variation (large wave numbers). For such modes the equations of shell theory may be taken in the form

$$
\begin{aligned}
&D\Delta\Delta w - \frac{1}{AB}\left(\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{1}{R_2}\frac{\partial\varphi}{\partial\alpha} + \frac{\partial}{\partial\beta}\frac{A}{B}\frac{1}{R_1}\frac{\partial\varphi}{\partial\beta}\right) = q \\
&\frac{1}{Eh}\Delta\Delta\varphi + \frac{1}{AB}\left(\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{1}{R_2}\frac{\partial w}{\partial\alpha} + \frac{\partial}{\partial\beta}\frac{A}{B}\frac{1}{R_1}\frac{\partial w}{\partial\beta}\right) = 0
\end{aligned}
\qquad (1.1)
$$

Here, w is the normal deflection, ϕ a function of the tangential forces, q the intensity of the normal load, E the elastic modulus, h the shell thickness, D the shell stiffness, A and B the Lamé coefficients of

oscillations the classical theory of plates and shells may be unsuitable and must be replaced by exact equations taking account of the effects of shear and rotary inertia.

the middle surface, R_1 and R_2 the radii of curvature corresponding to the lines β = const and α = const, and Δ is the Laplace operator for the middle surface

$$\Delta = \frac{1}{AB}\left(\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\frac{A}{B}\frac{\partial}{\partial\beta}\right)$$

We shall seek solutions of the system (1.1) for a certain region Ω bordering upon the line of distortion γ. We consider the dimensions of the region Ω to be small in order that changes in the metric of the middle surface inside this region may be neglected, thus setting $A \approx$ const, $B \approx$ const, $R_1 \approx$ const, $R_2 \approx$ const. At the same time the region Ω must be large enough so that many half-waves can be included (Fig. 1). These requirements will always be met by modes of oscillation with high values of the exponent of variation.

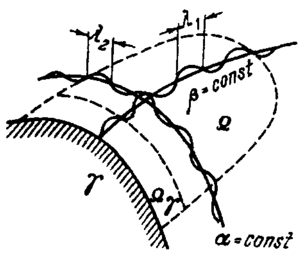

Fig. 1.

By the introduction of new variables $Ad\alpha = dx_1$, $Bd\beta = dx_2$, we obtain a system of equations

$$\begin{gathered} D\Delta\Delta w - \left(\frac{1}{R_2}\frac{\partial^2\varphi}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2\varphi}{\partial x_2^2}\right) + \frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2} = 0 \\ \frac{1}{Eh}\Delta\Delta\varphi + \left(\frac{1}{R_2}\frac{\partial^2 w}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2 w}{\partial x_2^2}\right) = 0 \end{gathered} \qquad (1.2)$$

which replaces (1.1).

The intensity of the normal inertia forces has replaced q in the first equation, with γ being the density of the shell material and g the acceleration of gravity; tangential components of the inertia forces may be neglected for modes of oscillation with high exponents of variation.

The substitution of expressions

$$w(x_1, x_2, t) = w_*(x_1, x_2)\exp(i\omega t)$$
$$\varphi(x_1, x_2, t) = \varphi_*(x_1, x_2)\exp(i\omega t)$$

into Equations (1.2), where $w_*(x_1, x_2)$ and $\phi_*(x_1, x_2)$ are the forms of oscillation and ω is the real frequency, leads to the equations

$$D\Delta\Delta w_* - \left(\frac{1}{R_2}\frac{\partial^2\varphi_*}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2\varphi_*}{\partial x_2^2}\right) - \frac{\gamma h\omega^2}{g} w_* = 0$$
$$\frac{1}{Eh}\Delta\Delta\varphi_* + \left(\frac{1}{R_2}\frac{\partial^2 w_*}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2 w_*}{\partial x_2^2}\right) = 0 \qquad (1.3)$$

The second equation will be satisfied if one sets

$$w_* = \Delta\Delta\psi_*, \qquad \varphi_* = -Eh\left(\frac{1}{R_2}\frac{\partial^2\psi_*}{\partial x_1^2} + \frac{1}{R_1}\frac{\partial^2\psi_*}{\partial x_2^2}\right) \qquad (1.4)$$

Then the first equation takes the form

$$\Delta\Delta\Delta\Delta\psi_* + \frac{Eh}{D}\left(\frac{1}{R_2^2}\frac{\partial^4\psi_*}{\partial x_1^4} + \frac{2}{R_1R_2}\frac{\partial^4\psi_*}{\partial x_1^2\partial x_2^2} + \frac{1}{R_1^2}\frac{\partial^4\psi_*}{\partial x_2^4}\right) - \frac{\gamma h\omega^2}{gD}\Delta\Delta\psi_* = 0 \qquad (1.5)$$

It is not difficult to see that

$$\psi_* = \psi_0 \sin k_1 (x_1 - x_1^\circ) \sin k_2 (x_2 - x_2^\circ) \qquad \left(k_1 = \frac{\pi}{\lambda_1},\ k_2 = \frac{\pi}{\lambda_2}\right) \qquad (1.6)$$

is a solution of Equation (1.5).

Here, λ_1 and λ_2 are half-wavelengths in the directions of β = const and α = const (Fig. 1), x_1° and x_2° are certain limiting phases, and ψ_0 is a normalizing constant. Under certain conditions, which will be established later, and for a certain choice of the limiting phases x_1° and x_2°, Expression (1.6) may be considered as an asymptotic expression for the forms of natural oscillation, applicable everywhere except in a region Ω_γ bordering upon the lines of distortion γ (it satisfies conditions on the lines of distortion only in the case where the lines of distortion represent a supported edge). The corresponding asymptotic expression for the frequency ω has the form

$$\omega^2 = \frac{gD}{\gamma h}\left[(k_1^2 + k_2^2)^2 + \frac{Eh}{D}\frac{(k_1^2 / R_2 + k_2^2 / R_1)^2}{(k_1^2 + k_2^2)^2}\right] \qquad (1.7)$$

We shall seek those solutions of Equation (1.5) which satisfy all conditions on the lines of distortion γ and which approach the solution (1.6) asymptotically as the internal region increases. If such solutions exist, then the dynamic edge effect exists also. In the opposite case we shall speak of a *degeneration* of the dynamic edge effect.

Assume that the line of distortion coincides with the line $x_1 = 0$. We seek a solution in the neighborhood of the line of distortion in the form

$$\psi_* (x_1, x_2) = \Psi (x_1) \sin k_2 (x_2 - x_2^\circ) \qquad (1.8)$$

where $\Psi (x_1)$ is a function as yet unknown. Substitution in (1.5) gives

$$\Psi^{\mathrm{VIII}} - 4k_2^2\Psi^{\mathrm{VI}} + 6k_2^4\Psi^{\mathrm{IV}} - 4k_2^6\Psi'' + k_2^8\Psi +$$
$$+\frac{Eh}{D}\left(\frac{1}{R_2^2}\Psi^{\mathrm{IV}} - \frac{2k_2^2}{R_1R_2}\Psi'' + \frac{k_2^4}{R_1^2}\Psi\right) - \frac{\gamma h\omega^2}{gD}(\Psi^{\mathrm{IV}} - 2k_2^2\Psi'' + k_2^4\Psi) = 0 \qquad (1.9)$$

The assumption that $\Psi = Ce^{sx_1}$, where C and s are constants, leads to the characteristic equation

$$\Delta(s^2) = s^8 - 4k_2^2 s^6 + \left(6k_2^4 + \frac{Eh}{R_2^2 D} - \frac{\gamma h\omega^2}{gD}\right)s^4 -$$
$$-\left(4k_2^6 + \frac{2Ehk_2^2}{R_1R_2D} - 2k_2^2\frac{\gamma h\omega^2}{gD}\right)s^2 + k_2^8 + \frac{k_2^4 Eh}{R_1^2 D} - k_2^4\frac{\gamma h\omega^2}{gD} = 0 \qquad (1.10)$$

If the frequency ω is determined by Formula (1.7), then it follows from (1.6) that Expression (1.10) contains two pure imaginary roots $s_{1,2} = \pm ik_1$. These roots may be conveniently separated out. By means of this separation we can convince ourselves that

$$\Delta(s^2) = \Delta_1(s^2)(s^2 + k_1^2)$$

where

$$\Delta_1(s^2) = s^6 - (k_1^2 + 4k_2^2)s^4 +$$
$$+ k_2^2\left\{2k_1^2 + 5k_2^2 - \frac{Eh}{D(k_1^2+k_2^2)^2}\left[2k_1^2\left(\frac{1}{R_1R_2} - \frac{1}{R_2^2}\right) + k_2^2\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)\right]\right\}s^2 -$$
$$- k_2^4\left\{k_1^2 + 2k_2^2 - \frac{Eh}{D(k_1^2+k_2^2)^2}\left[k_1^2\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right) + 2k_2^2\left(\frac{1}{R_1^2} - \frac{1}{R_1R_2}\right)\right]\right\} = 0 \quad (1.11)$$

Equation (1.11) is the basic equation which determines completely the properties of the dynamic edge effect for oscillations having the wave numbers k_1 and k_2.

2. Classification of the types of dynamic edge effect. (a) Assume that all the roots s^2 of Equation (1.11) are different, real and positive. Then among the characteristic exponents s appear three different real negative values $s_3 = -\alpha_1$, $s_4 = -\alpha_2$, $s_5 = -\alpha_3$. By the rejection of terms increasing with x_1 in the general integral of (1.9) we obtain

$$\Psi(x_1) = C_1\sin k_1x_1 + C_2\cos k_1x_1 + C_3e^{-\alpha_1x_1} + C_4e^{-\alpha_2x_1} + C_5e^{-\alpha_3x_1} \quad (2.1)$$

With this expression we may satisfy all four conditions on the edge $x_1 = 0$. A fifth condition may be taken as a normalizing condition or as an initial condition. If the solution (1.6) for the internal region is normalized, then as the fifth condition we take the limiting relation

$$\lim_{x_1\to\infty}\Psi(x_1) = \psi_0\sin k_1(x_1 - x_1^\circ) \qquad (2.2)$$

The limiting phase x_1° may be found from this condition.

Thus, if all the roots s^2 are different, real and positive, then the solution for a dynamic edge effect exists and is described by the last three terms of Expression (2.1). Since these terms decay and do not oscillate, we call this effect *nonoscillatory*.

b) For multiple real positive roots the edge effect remains nonoscillatory. Thus, if $\alpha_2 = \alpha_3$, then the solution for the region bordering on the edge $x_1 = 0$ has the form

$$\Psi(x_1) = C_1 \sin k_1 x_1 + C_2 \cos k_1 x_1 + C_3 e^{-\alpha_1 x_1} + C_4 e^{-\alpha_2 x_1} + x_1 C_5 e^{-\alpha_2 x_1} \quad (2.3)$$

c) Assume that Equation (1.11) has one real positive root and two complex conjugate roots. Then among the characteristic exponents three exponents will be found which describe the dynamic edge effect: $s_3 = -\alpha_1$, $s_4 = -\alpha_2 + i\beta_2$, $s_5 = -\alpha_2 - i\beta_2$. The solution for a bounded region takes the form

$$\Psi(x_1) = C_1 \sin k_1 x_1 + C_2 \cos k_1 x_1 + C_3 e^{-\alpha_1 x_1} + C_4 e^{-\alpha_2 x_1} \sin \beta_2 x_1 + C_5 e^{-\alpha_2 x_1} \cos \beta_2 x_1 \quad (2.4)$$

We call this edge effect *oscillatory*.

d) If the equation $\Delta_1(s^2) = 0$ has only one negative or zero root, then among the exponents $s_1, s_2, \ldots, s_8$ there will not be found three with real negative values. Thus, a solution cannot be constructed having the property (2.2) and be sufficiently arbitrary so as to satisfy the four conditions on the line $x_1 = 0$. By analogy with the static edge effect [3] we speak of this as a *degeneration* of the dynamic edge effect.

3. Certain special cases. We shall dwell on certain special cases which permit results to be obtained easily.

a) The dynamic edge effect in plates. If $R_1 \to \infty$, $R_2 \to \infty$, then Equation (1.11) simplifies and reduces to a product

$$\Delta_1(s^2) = (s^2 - k_1^2 - 2k_2^2)(s^4 - 2k_2^2 + k_2^4) = 0 \quad (3.1)$$

This corresponds to a separation of the system (1.3) into two equations

$$D\Delta\Delta w_* - \frac{\gamma h \omega^2}{g} w_* = 0, \qquad \Delta\Delta\varphi_* = 0$$

The second equation describes the plane state of stress in the plate so that the corresponding roots of Equation (3.1) must be rejected. The solution for the bounded region is of the form

$$\Psi(x_1) = C_1 \sin k_1 x_1 + C_2 \cos k_1 x_1 + C_3 \exp[-x_1 (k_1^2 + 2k_2^2)^{1/2}]$$

Three constants C_1, C_2 and C_3 are sufficient to satisfy two conditions on the lines of distortion and the limiting condition (2.2). For the normal deflection $w_*(x_1, x_2)$ we obtain the expression

$$w_* (x_1, x_2) = W (x_1) \sin k_2 (x_2 - x_2^\circ) \tag{3.2}$$

where $W(x_1)$ is found from the formula

$$W (x_1) = \Psi^{\mathrm{IV}} - 2k_2^2\Psi'' + k_2^4\Psi \tag{3.3}$$

corresponding to the first of relations (1.4).

Thus, we arrive at the formula

$$W (x_1) = C_1^* \sin k_1x_1 + C_2^* \cos k_1x_1 + C_3^* \exp [-x_1 (k_1^2 + 2k_2^2)^{1/2}] \tag{3.4}$$

Here, C_1*, C_2* and C_3* are certain new constants. Thus, we have a nonoscillatory edge effect in plates.

b) Analog of the simple edge effect. Assume that $k_1^2 \gg k_2^2$ near the edge $x_1 = 0$. This signifies that the exponent of variation in the direction α = const is small by comparison with the exponent of variation in the direction β = const. Under this condition Equation (1.11) may be replaced by the approximate equation

$$\Delta_1 (s^2) = s^6 - k_1^2s^4 = 0$$

Discarding the four zero roots and one positive root, the solution is represented in the form

$$\Psi (x_1) = C_1 \sin k_1x_1 + C_2 \cos k_1x_1 + C_3e^{-k_1x_1} \tag{3.5}$$

It is not difficult to show that this solution is analogous to the simple edge effect in statics of shells. We start with the equation

$$D \frac{\partial^4 w}{\partial x_1^4} + \frac{Eh}{R_2^2} w + \frac{\gamma h}{g} \frac{\partial^2 w}{\partial t^2} = 0$$

on the assumption that all conditions of applicability are satisfied [3]. A substitution $w(x_1, t) = W(x_1) \exp (i\omega t)$ leads to the equation

$$W^{\mathrm{IV}} + \frac{Eh}{R_2^2D} W - \frac{\gamma h\omega^2}{gD} W = 0$$

As asymptotic expressions for the mode of oscillation and for the frequency we take, as usual

$$W (x_1) = w_0 \sin k_1 (x_1 - x_1^\circ), \qquad \omega^2 = \frac{gD}{\gamma h}\left(k_1^4 + \frac{Eh}{R_2^2D}\right)$$

The characteristic equation

$$\Delta(s^2) = s^4 + \frac{Eh}{R_2^2 D} - \frac{\gamma h \omega^2}{gD} = 0$$

has roots $s_{1,2} = \pm ik_1$, $s_{3,4} = \pm k_1$. Thus, once more we are led to a solution of the type of (3.5).

The result that the curvature shows no effect on the dynamic edge effect is not unexpected. The requirement $k_1^2 \gg k_2^2$, in combination with the condition that k_2 is sufficiently large, leads to the shell behaving practically as a plate. We note that the solution is not as valuable in the dynamic problem as its static analog; in contrast to the static problem it is completely contained in the solution (3.4) for $k_1^2 \gg k_2^2$.

c) Spherical shell. If $R_1 = R_2$, then we are once more let to Equation (3.1). In contrast to the plate, the roots of the equation $s^4 - 2k_2^2 s^2 + k_2^4 = 0$ are not superfluous here. The solution takes the form of (2.3):

$$\Psi(x_1) = C_1 \sin k_1 x_1 + C_2 \cos k_1 x_1 + C_3 \exp\left[-x_1 (k_1^2 + 2k_2^2)^{1/2}\right] + \\ + C_4 \exp(-k_2 x_1) + x_1 C_5 \exp(-k_2 x_1) \qquad (3.6)$$

Thus, we have a nonoscillating edge effect with multiple roots. It is worth noting that the radius of curvature does not enter into the solution (3.6).

4. General case. Existence condition for the edge effect. Consider the most general equation (1.11) into which we introduce the nondimensional parameters

$$z_1 = k_1 \left(\frac{DR_1^2}{Eh}\right)^{1/4}, \quad z_2 = k_2 \left(\frac{DR_1^2}{Eh}\right)^{1/4}, \quad s_* = s \left(\frac{DR_1^2}{Eh}\right)^{1/4}, \quad \chi = \frac{R_1}{R_2} \qquad (4.1)$$

The equation takes the form

$$s_*^6 - (z_1^2 + 4z_2^2) s_*^4 + z_2^2 \left[2z_1^2 + 5z_2^2 - \frac{2z_1^2 \chi (1-\chi) + z_2^2 (1-\chi^2)}{(z_1^2 + z_2^2)^2}\right] s_*^2 - \\ - z_2^4 \left[z_1^2 + 2z_2^2 - \frac{z_1^2 (1-\chi^2) + 2z_2^2 (1-\chi)}{(z_1^2 + z_2^2)^2}\right] = 0 \qquad (4.2)$$

It has been shown that a solution for a type of dynamic edge effect may be constructed if Equation (4.2) has no negative or zero roots for s_*^2. According to a well-known theorem of Descartes, Equation (4.2) has only positive or complex conjugate roots for s_*^2 if the coefficients exhibit a regular alternation of sign in sequence. In this case, the relevant condition is that for the free negative term

$$z_1^2 + 2z_2^2 - \frac{z_1^2 (1-\chi^2) + 2z_2^2 (1-\chi)}{(z_1^2 + z_2^2)^2} > 0$$

Consider this condition in more detail. We note that it is satisfied for all values of z_1 and z_2 if $\chi \geqslant 1$. Consequently, for $R_1 \geqslant R_2$ the dynamic edge effect always exists. In particular, it will always hold in shells of zero Gaussian curvature near a nonasymptotic edge.

If $\chi < 1$, then the edge effect exists only for sufficiently high exponents of variation. We introduce polar coordinates

$$z_1 = r\cos\varphi, \qquad z_2 = r\sin\varphi$$

The edge effect is not degenerate if

$$r^4 > \frac{1-\chi^2+(1-\chi)^2\sin^2\varphi}{1+\sin^2\varphi} \qquad (4.3)$$

Thus, for an asymptotic edge in a shell of zero Gaussian curvature we have the condition $r > 1$, or

$$k_1^2 + k_2^2 > \sqrt{\frac{Eh}{DR_1^2}} \qquad (4.4)$$

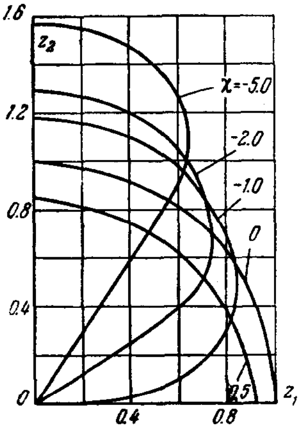

Fig. 2.

The region of degeneration for different values of the ratio $\chi = R_1/R_2$ is shown in Fig. 2.

As an example, consider a cylindrical panel with a square shape and sides of length a. For such a panel $k_1 = m\pi/a$, $k_2 = n\pi/a$, where m and n are not, in general, integers. Condition (4.4) becomes

$$m^2 + n^2 > \frac{a^2}{\pi^2 hR}\sqrt{12(1-\mu^2)}$$

where μ is Poisson's ratio. If $a/h = 100$, $R/a = 10$, $\mu = 0.25$, then we must have $m^2 + n^2 = 3.41$ for a dynamic edge effect to exist. This condition is fulfilled for $m = 2$, $n = 1$. If $a/h = 100$, $R/a = 1$, then we must have $m^2 + n^2 = 34.1$. We note that the static edge effect at an asymptotic edge for a shell of zero Gaussian curvature is always degenerate.

For a shell of negative Gaussian curvature the region of degeneracy stretches along the z_2-axis (Fig. 2) with increasing $|\chi|$ and vanishes as $\chi \to -\infty$.

5. General case. Type of dynamic edge effect. By introduction of the variables

$$s_*^2 = s_{**}^2 + \tfrac{1}{3}(z_1^2 + 4z_2^2) \qquad (5.1)$$

into Equation (4.2) we obtain

$$s_{**}^6 + 3ps_{**}^2 + 2q = 0 \tag{5.2}$$

where

$$\begin{aligned} p &= -\tfrac{1}{9}\{r^4 + 3\sin^2\varphi\,[2\chi(1-\chi) + (1-\chi)^2\sin^2\varphi]\} \\ q &= -\tfrac{1}{27}\{r^6 + 9r^2\sin^2\varphi\,[\chi(1-\chi) - (1-\chi)^2\sin^2\varphi]\} \end{aligned} \tag{5.3}$$

The discriminant of Equation (5.2) has the form

$$p^3 + q^2 = -\tfrac{1}{27}\sin^4\varphi\,(1-\chi)^2 f(r) \tag{5.4}$$

in which

$$\begin{aligned} f(r) = r^8 + r^4\,[\chi^2 + 10\chi(1-\chi)\sin^2\varphi - 2(1-\chi)^2\sin^4\varphi] + \\ + (1-\chi)\,[2\chi - (1-\chi)\sin^2\varphi]^3\sin^2\varphi \end{aligned} \tag{5.5}$$

The edge effect oscillates if $p^3 + q^2 > 0$ and does not oscillate if $p^3 + q^2 < 0$. If $p^3 + q^2 = 0$, the solution has the form of (2.3). Condition (4.3) is obviously fulfilled.

Certain general statements may be made without difficulty from a study of Equations (5.4) and (5.5). If $r \to \infty$, then the sign of $f(r)$ is determined by the sign of the first term and consequently $p^3 + q^2 < 0$. This result is quite natural* when one considers that for large exponents of variation a shell behaves practically as a plate.

The edge effect may be shown to be oscillatory for small r. This is true for shells of positive as well as negative Gaussian curvature. Only for $0 < \chi < 1$ is the edge effect nonoscillatory for all values of r. Indeed, in this region Equation (5.5) does not, in general, have positive real roots.

The behavior of the edge effect for different ratios $\chi = R_1/R_2$ is shown in Fig. 3 as a frontal dimetric projection. The region of oscillating edge effect is shown by the light shading and the region of degeneracy by the dense shading.

Equation (4.2) is not suited to a study of the edge effect near a nonasymptotic edge for a shell of zero Gaussian curvature; in this case $R_1 \to \infty$. By introduction of the nondimensional parameters

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* Excluding the case of the spherical shell ($\chi = 1$) and the analogy of the simple edge effect ($\sin\phi = 0$) for which $p^3 + q^2 = 0$ for all r.

$$\zeta_1 = k_1\left(\frac{DR_2^2}{Eh}\right)^{1/4}, \qquad \zeta_2 = k_2\left(\frac{DR_2^2}{Eh}\right)^{1/4}, \qquad s_{**}^2 = s^2\left(\frac{DR_2^2}{Eh}\right)^{1/2} - \frac{1}{3}(\zeta_1^2 + 4\zeta_2^2)$$

and passage to polar coordinates $\zeta_1 = \rho\cos\phi$, $\zeta_2 = \rho\sin\phi$, we obtain Equation (5.2) with the coefficients

$$p = -\frac{1}{9}\left[\rho^4 - 3\sin^2\varphi\,(2 - \sin^2\varphi)\right]$$

$$q = -\frac{1}{27}\left[\rho^6 - 9\rho^2\sin^2\varphi\,(1 + \sin^2\varphi)\right]$$

The discriminant

$$p^3 + q^2 = -\frac{1}{27}\sin^4\varphi\left[\rho^8 + \rho^4(1 - 10\sin^2\varphi - 2\sin^4\varphi) - \sin^2\varphi(2 - \sin^2\varphi)^3\right]$$

is positive if $\rho < \rho_0$, where

$$2\rho_0^4 = -1 + 10\sin^2\varphi + 2\sin^4\varphi + (1 + 4\sin^2\varphi)^{3/2} \qquad (5.6)$$

The region determined by the inequality $\rho < \rho_0$ is shown shaded in Fig. 4.

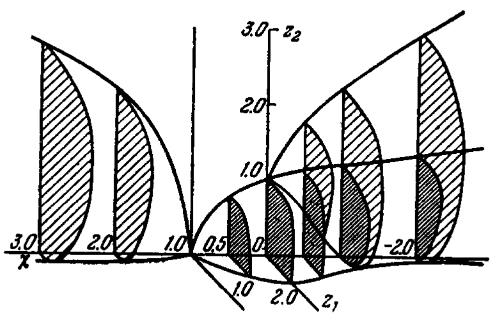

Fig. 3.

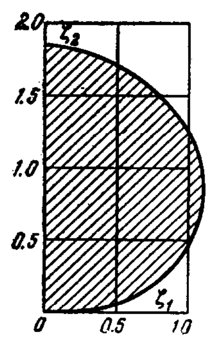

Fig. 4.

For a closed circular cylindrical shell of radius R and length a we have $k_1 = m\pi/a$, $k_2 = n/R$, where m and n are wave numbers. The condition $\rho < \rho_0$ takes the form

$$m^2\left(\frac{\pi R}{a}\right)^2 + n^2 < \frac{R}{h}\rho_0^2\sqrt{12(1 - \mu^2)} \quad (5.7)$$

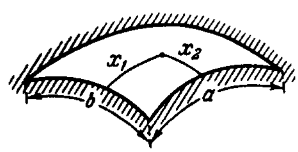

Fig. 5.

If this condition is fulfilled, the edge effect oscillates. For example, let $\pi R = a$, $R/h = 100$; then we have from (5.7)

$$m^2 + n^2 < 336\,\rho_0^2$$

This condition is satisfied for very high frequencies if only n is not too small.

6. Some applications. A series of applications of the theory of the dynamic edge effect in plates was considered in [1] and [2]. Here, the theory is applied to the study of frequency spectra and the modes of natural oscillation for spherical panels clamped at the edges and bounded by four orthogonal lines of curvature (Fig. 5).

The oscillations of such a shell are described by Equation (1.5) for $R_1 = R_2 = R$ and the boundary conditions

$$\begin{aligned} u_* = v_* = w_* = \frac{\partial w_*}{\partial x_1} = 0 \qquad (x_1 = 0 \text{ and } x_1 = a) \\ u_* = v_* = w_* = \frac{\partial w_*}{\partial x_2} = 0 \qquad (x_2 = 0 \text{ and } x_2 = b) \end{aligned} \tag{6.1}$$

Here u_* and v_* are tangential displacements in the direction of $x_2 =$ const and $x_1 =$ const, respectively.

Consider first a solution near the edge $x_1 = 0$. We have, by analogy with Expression (1.8)

$$\begin{gathered} u_* (x_1, x_2) = U(x_1) \sin k_2 (x_2 - x_2^\circ), \qquad v_* (x_1, x_2) = V(x_1) \cos k_2 (x_2 - x_2^\circ) \\ w_* (x_1, x_2) = W(x_1) \sin k_2 (x_2 - x_2^\circ) \end{gathered} \tag{6.2}$$

By the use of Formulas (3.3) and (3.6) we obtain

$$W(x_1) = C_1^* \sin k_1 x_1 + C_2^* \cos k_1 x_1 + C_3^* \exp [-x_1 (k_1^2 + 2k_2^2)^{1/2}] \tag{6.3}$$

where $C_j^* = (k_1^2 + k_2^2)^2 C_j$. Terms containing two integration constants drop out of the general solution (3.6). Nevertheless, these two constants appear again for the determination of the tangential displacements u_* and v_*.

We start from the system of equations in terms of displacements equivalent in their simplifying assumptions to the system (1.3):

$$\begin{gathered} \frac{\partial^2 u_*}{\partial x_1^2} + \frac{1-\mu}{2}\frac{\partial^2 u_*}{\partial x_2^2} + \frac{1+\mu}{2}\frac{\partial^2 v_*}{\partial x_1 \partial x_2} - \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right)\frac{\partial w_*}{\partial x_1} = 0 \\ \frac{1+\mu}{2}\frac{\partial^2 u_*}{\partial x_1 \partial x_2} + \frac{\partial^2 v_*}{\partial x_2^2} + \frac{1-\mu}{2}\frac{\partial^2 v_*}{\partial x_1^2} - \left(\frac{1}{R_2} + \frac{\mu}{R_1}\right)\frac{\partial w_*}{\partial x_2} = 0 \\ -\left(\frac{1}{R_1} + \frac{\mu}{R_2}\right)\frac{\partial u_*}{\partial x_1} - \left(\frac{1}{R_2} + \frac{\mu}{R_1}\right)\frac{\partial v_*}{\partial x_2} + \left(\frac{1}{R_1^2} + \frac{2\mu}{R_1 R_2} + \frac{1}{R_2^2}\right) w_* + \\ + \frac{h^2}{12}\Delta\Delta w_* - \frac{1-\mu^2}{Eh}\frac{\gamma h \omega^2}{g} w_* = 0 \end{gathered} \tag{6.4}$$

Upon substitution of (6.2), we obtain for $R_1 = R_2 = R$

$$U'' - \frac{1-\mu}{2} k_2^2 U - \frac{1+\mu}{2} k_2 V' = \frac{1+\mu}{R} W'$$
$$\frac{1+\mu}{2} k_2 U' - k_2^2 V + \frac{1-\mu}{2} V'' = \frac{1+\mu}{R} k_2 W \tag{6.5}$$
$$-\frac{1+\mu}{R} U' + \frac{1+\mu}{R} k_2 V =$$
$$= -\frac{2(1+\mu)}{R^2} W - \frac{h^2}{12} (W^{IV} - 2k_2^2 W'' + k_2^4 W) + \frac{1-\mu^2}{Eh} \frac{\gamma h \omega^2}{g} W$$

For the determination of U and V, any two equations may be taken from the system (6.5), for example the first two. It is easy to see that the characteristic equation of a system consisting of the first two equations has two pairs of multiple roots equal to $\pm k_2$. Therefore, for the construction of the solution for this system exhibiting the properties of the edge effect, two terms of the type lost from the solution (3.6) must be added to the particular solution $U_0(x_1)$ and $V_0(x_1)$. The number of integration constants, therefore, still stands at five; i.e. the edge effect will be completely determinate. Actual calculations result in the formulas

$$U(x_1) = U_0(x_1) + C_4 e^{-k_2 x_1} + x_1 C_5 e^{-k_2 x_1}$$
$$V(x_1) = V_0(x_1) - \left(C_4 - \frac{3-\mu}{1+\mu} \frac{C_5}{k_2}\right) e^{-k_2 x_1} - x_1 C_5 e^{-k_2 x_1} \tag{6.6}$$

It follows from the preceding discussion that the tangential edge conditions do not affect the normal deflection w_*. This permits a simple finding of an asymptotic estimate of the natural frequencies and wave numbers.

Upon application of the conditions $W(0) = W'(0) = 0$, we obtain from Expression (6.3)

$$W(x_1) = \sin k_1 x_1 - \frac{k_1}{(k_1^2 + 2k_2^2)^{1/2}} \{\cos k_1 x_1 - \exp[-x_1 (k_1^2 + 2k_2^2)^{1/2}]\} \tag{6.7}$$

and analogously for the edge $x_1 = a$

$$W(x_1) = c \sin k_1 (a - x_1) - \frac{ck_1}{(k_1^2 + 2k_2^2)^{1/2}} \{\cos k_1 (a - x_1) -$$
$$- \exp[-(a - x_1)(k_1^2 + 2k_2^2)^{1/2}]\}$$

where c is a certain new constant.

We now require that both solutions coincide in the interior of the region. It appears that this requirement may be met only with an error of the order of

$$\varepsilon \sim \exp\left[-\tfrac{1}{2}\, a\, (k_1^2 + 2k_2^2)^{1/2}\right] \tag{6.8}$$

We present the results without dwelling on the calculations. The solutions separate into symmetrical ones with respect to the middle surface of the shell ($x_1 = a/2$)

$$c = 1, \qquad \cot\frac{k_1 a}{2} = -\frac{k_1}{(k_1^2 + 2k_2^2)^{1/2}} \tag{6.9}$$

and antisymmetrical solutions

$$c = -1, \quad \tan\frac{k_1 a}{2} = \frac{k_1}{(k_1^2 + 2k_2^2)^{1/2}} \tag{6.10}$$

The parameter k_1 may be found for a given k_2 from Equations (6.9) and (6.10).

We obtain analogous results by considering the dynamic edge effect at the edges $x_2 = 0$ and $x_2 = b$, together with the "pieced-together" solution for the internal region. A closed system of equations in k_1 and k_2 results.

The modes of the free oscillations fall into four groups according to the type of symmetry. For the first type (symmetry in both directions) we have the system

$$\cot\frac{k_1 a}{2} = -\frac{k_1}{(k_1^2 + 2k_2^2)^{1/2}}, \qquad \cot\frac{k_2 b}{2} = -\frac{k_2}{(k_2^2 + 2k_1^2)^{1/2}} \tag{6.11}$$

For antisymmetrical modes in both directions we have

$$\tan\frac{k_1 a}{2} = \frac{k_1}{(k_1^2 + 2k_2^2)^{1/2}}, \qquad \tan\frac{k_2 b}{2} = \frac{k_2}{(k_2^2 + 2k_1^2)^{1/2}} \tag{6.12}$$

We have two mixed types combining one equation from (6.11) and (6.12). The ratio $\nu = k_2/k_1$ may be found from the equation

$$\nu = \frac{a}{b}\,\frac{\tan^{-1}(1 - 2\nu^{-2})^{1/2} + \tfrac{1}{2}n\pi}{\tan^{-1}(1 + \nu^2)^{1/2} + \tfrac{1}{2}m\pi} \tag{6.13}$$

where the principal values of the inverse trigonometric functions are to be considered, and where m and n are positive integers (wave numbers).

Equation (6.13) is easily solved by the method of successive approximations, in which the zeroth approximation may be taken as the asymptotic value $\nu = na/mb$.

Results calculated for the case $a = b$ are given in the table, where ∞ is the coefficient in the frequency formula (1.7)

$$\omega^2 = \frac{gD}{\gamma h a^4}\left(\pi^4\alpha^2 + \frac{Eh}{DR^2}\right)$$

Note that for the modes of oscillation for which $k_1 = k_2$, Formulas (6.11) and (6.12) give

$$\lambda_1 = \lambda_2 = \frac{a}{m + {}^1/_3} \qquad (m = 1, 2, \ldots)$$

We find from this that $\alpha = 2(m + 1/3)^2$.

TABLE

m	n	ν	a/λ_1	a/λ_2	α		Percentage difference
					Present method	Iguchi solution	
1	1	1.0000	4/3	4/3	3.556	3.646	2.53
2	1	2.0265	2.4372	1.2027	7.386	7.437	0.69
2	2	2.0000	7/3	7/3	10.889	10.965	0.70
3	1	3.0377	3.4688	1.1420	13.336	13.395	0.42
3	2	1.5079	3.4012	2.2556	16.656	16.717	0.37
3	3	1.0000	10/3	10/3	22.222	—	—
4	1	4.0432	4.4816	1.1084	21.313	—	—
4	2	2.0132	4.4366	2.2038	24.540	24.631	0.36
4	3	1.3370	4.3832	3.2784	29.960	—	—
4	4	1.0000	13/3	13/3	37.556	—	—

It is well known that this problem has no exact solution. For the clamped plate Iguchi [4] has a sufficiently accurate solution in a series of functions satisfying all edge conditions and limited to six terms. Evidently his value of the fundamental frequency may be considered as exact. The last columns of the table give a comparison with the results of Iguchi. The table shows that the difference between the results for $m = n = 1$ is not great. This confirms the idea that the solutions for the dynamic edge effect satisfy Equations (1.3) exactly and all edge conditions. Any error is due to the "pieced-together" solution constructed for two opposite edges, an error of the order estimated by Formula (6.8). By substitution of the values $k_1 = k_2 = 4\pi/3a$ in (6.8), we find $\epsilon \sim \exp(-2\pi/\sqrt{3}) \approx 0.027$, which is close to the difference shown in the last column of the table. As one might guess from the nature of the simplification, the method gives natural frequencies which are too low. The error for the higher frequencies will be less than that shown in the table, since the Iguchi method gives upper approximations.

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Translated by E.Z.S.